On the $mathcal{R}$-Boundedness of Solution Operators for the weak Dirichlet-Neumann Problem (Mathematical Analysis of Incompressible Flow)

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On the $\mathcal{R}$-Boundedness of Solution Operators for the weak Dirichlet-Neumann Problem

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1 Introduction

We consider an autonomous evolution equation:

$$u_t - Au = f, \ \ Bu = g \ \text{ for } t > 0, \ \ u|_{t=0} = u_0. \tag{1.1}$$

Here, A and B are some linear operators and $Bu = g$ represents a non-homogeneous boundary condition. Through the Laplace transform with respect to time variable, we have the corresponding generalized resolvent problem:

$$\lambda v - Av = f, \ \ Bv = g. \tag{1.2}$$

Here, the reason why we call (1.2) a generalized resolvent problem is that we consider non-homogeneous boundary condition. Let v be represented by $v = R(\lambda)(f, g)$ with some solution operator $R(\lambda)$ to (1.2). When $g = 0$, if $R(\lambda)$ satisfies the condition of Hille-Yosida type, then A generates a continuous semigroup, which gives us a unique solution to the Cauchy problem:

$$u_t - Au = 0, \ \ Bu = 0 \ \text{ for } t > 0, \ \ u|_{t=0} = u_0.$$

Moreover, if $R(\lambda)$ satisfies suitable multiplier conditions, the Laplace inverse transform of $R(\lambda)(f, g)$ gives us a solution to the non-homogeneous initial-boundary value problem:

$$u_t - Au = f, \ \ Bu = g \ \text{ for } t \in \mathbb{R}.$$

In addition, the condition f and g vanish for $t < 0$ implies that u also vanishes for $t < 0$, which especially means that $u|_{t=0} = 0$. Combining these two results, we can solve (1.1). In fact, Sakamoto [6] proved the unique existence of solutions to the initial-boundary mixed problem for the general hyperbolic equations with the boundary condition satisfying uniform Lopatinski conditions in rather general domains *. Since her problem is hyperbolic, she considered the problem the in the L_2 framework. Therefore, the boundedness of the operator norm of $R(\lambda)$ implies the unique existence and suitable estimates of solutions to the evolution equations by means of the Plancherel formula.

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*Kreiss [5] treated the hyperbolic system case, but he proved *a priori* estimates only.

We wanted to extend Sakamoto's approach to the L_p framework for a long time and the Weis theorem [10] of the L_p-boundedness ($1 < p < \infty$) of the operator valued Fourier multiplier theorem enables us to extend Sakamoto's approach at least to the parabolic type equations including Stokes equations for both of the compressible and incompressible fluid flows (cf. Enomoto and Shibata [4] and Shibata [8]). In fact, the $\mathcal{R}$-boundedness of solution operator $R(\lambda)$ implies not only the generation of analytic semigroup but also L_p-L_q maximal regularity by means of the Weis theorem.

In this paper, we explain how to prove the $\mathcal{R}$-boundedness of solution operators by treating the following generalized resolvent problem for the weak Dirichlet-Neumann problem:

$$(\lambda u, \varphi)_\Omega + (\nabla u, \nabla\varphi)_\Omega = -(f, \nabla\varphi)_\Omega + (g, \varphi)_\Omega + < h_n, \varphi >_{\Gamma_2} \quad \text{for any } \varphi \in W^1_{q,\Gamma_1}(\Omega), \tag{1.3}$$

subject to $u = h_d$ on Γ_1. Here, Ω is a uniform C^1 domain in $\mathbb{R}^N$ ($N \geq 2$) with boundary $\Gamma_1 \cup \Gamma_2$. We assume that $\Gamma_1 \cap \Gamma_2 = \emptyset$. For any domain G in $\mathbb{R}^N$, we set $(a, b)_G = \int_G a(x)b(x)\,dx$. When Γ is a C^1 hypersurface with surface element $d\sigma$, we set $< a, b >_\Gamma = \int_\Gamma a(x)b(x)\,d\sigma$. $W^1_{q,\Gamma_1}(\Omega)$ denotes the functional space: $\{\varphi \in W^1_q(\Omega) \mid \varphi|_{\Gamma_1} = 0\}$.

Before stating our main results, first we introduce the Weis operator valued Fourier multiplier theorem. For this purpose, we introduce the notion of $\mathcal{R}$ boundedness of operator families.

Definition 1.1. Let X and Y be two Banach spaces and $\mathcal{L}(X, Y)$ denotes the set of all bounded linear operators from X into Y. A family of operators $\mathcal{T} \subset \mathcal{L}(X, Y)$ is called $\mathcal{R}$ bounded, if there exist constants $C > 0$ and $p \in [1, \infty)$ such that for any natural number n, $\{T_j\}_{j=1}^n \subset \mathcal{T}$, $\{x_j\}_{j=1}^n \subset X$ and sequences $\{r_j(u)\}_{j=1}^n$ of independent, symmetric, $\{-1, 1\}$-valued random variables on $[0, 1]$ there holds the inequality:

$$\Big\{\int_0^1 \|\sum_{j=1}^n r_j(u)T_jx_j\|_Y^p\,du\Big\}^{\frac{1}{p}} \leq C\Big\{\int_0^1 \|\sum_{j=1}^n r_j(u)x_j\|_X^p\,du\Big\}^{\frac{1}{p}}.$$

The smallest such C is called $\mathcal{R}$-bound of $\mathcal{T}$, which is denoted by $\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T})$.

Let $\mathcal{D}(\mathbb{R}, X)$ and $\mathcal{S}(\mathbb{R}, X)$ be the set of all X valued C^∞ functions having compact supports and the Schwartz space of rapidly decreasing X valued functions, respectively while $\mathcal{S}'(\mathbb{R}, X) = \mathcal{L}(\mathcal{S}(\mathbb{R}, \mathbb{C}), X)$, $\mathbb{C}$ being the set of all complex numbers. Given $M \in L_{1,\mathrm{loc}}(\mathbb{R} \setminus \{0\}, X)$, we define the operator $T_M : \mathcal{F}^{-1}\mathcal{D}(\mathbb{R}, X) \to \mathcal{S}'(\mathbb{R}, Y)$ by

$$T_M\phi = \mathcal{F}^{-1}[M\mathcal{F}[\phi]], \quad (\mathcal{F}[\phi] \in \mathcal{D}(\mathbb{R}, X)), \tag{1.4}$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and the Fourier inverse transform, respectively. The following theorem is obtained by Weis [10].

Theorem 1.2. *Let X and Y be two UMD Banach spaces and $1 < p < \infty$. Let M be a function in $C^1(\mathbb{R} \setminus \{0\}, \mathcal{L}(X, Y))$ such that*

$$\mathcal{R}_{\mathcal{L}(X,Y)}(\{(\tau\frac{d}{d\tau})^\ell M(\tau) \mid \tau \in \mathbb{R} \setminus \{0\}\} \leq \kappa < \infty \quad (\ell = 0, 1)$$

with some constant κ. Then, the operator T_M defined in (1.4) is extended to a bounded linear operator from $L_p(\mathbb{R}, X)$ into $L_p(\mathbb{R}, Y)$. Moreover, denoting this extension by T_M, we have

$$\|T_M\|_{\mathcal{L}(L_p(\mathbb{R},X), L_p(\mathbb{R},Y))} \leq C\kappa$$

for some positive constant C depending on p, X and Y.

Remark 1.3. For the definition of UMD space, we refer to a book due to Amann [1]. And, for $1 < q < \infty$, Lebesgue space $L_q(\Omega)$ and Sobolev space $W^m_q(\Omega)$ are both UMD spaces.

Secondly, we introduce the definition of uniform C^1 domains.

Definition 1.4. Let Ω be a domain in $\mathbb{R}^N$ with boundary $\partial\Omega$. We say that Ω is a uniform C^1 domain if there exist positive constants α, β and K such that for any $x_0 = (x_{01}, \dots, x_{0N}) \in \partial\Omega$ there exist a coordinate number j and a C^1 function $h(x')$ $(x' = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_N))$ defined on $B'_\alpha(x'_0)$ with $x'_0 = (x_{01}, \dots, x_{0j-1}, x_{0j+1}, \dots, x_{0N})$ and $\|h\|_{W^1_\infty(B'_\alpha(x'_0))} \le K$ such that

$$\Omega \cap B_\beta(x_0) = \{x \in \mathbb{R}^N \mid x_j > h(x') \ \ (x' \in B'_\alpha(x'_0))\} \cap B_\beta(x_0),$$
$$\partial\Omega \cap B_\beta(x_0) = \{x \in \mathbb{R}^N \mid x_j = h(x') \ \ (x' \in B'_\alpha(x'_0))\} \cap B_\beta(x_0).$$

Here, $B'_\alpha(x'_0) = \{x' \in \mathbb{R}^{N-1} \mid |x' - x'_0| < \alpha\}$, $\ B_\beta(x_0) = \{x \in \mathbb{R}^N \mid |x - x_0| < \beta\}$.

Thirdly, we recall some further symbols used throughout the paper. For any multi-index $\alpha = (\alpha_1, \dots, \alpha_N)$, we set $D^\alpha h = \partial_1^{\alpha_1} \cdots \partial_N^{\alpha_N} h$. We write $\nabla u = (D_1 u, \dots, D_N u)$ with $D_j = \partial/\partial x_j$. For any domain G in $\mathbb{R}^N$, $L_q(G)$ and $W^m_q(G)$ denote the usual Lebesgue space and Sobolev space, respectively, while $\|\cdot\|_{L_q(G)}$ and $\|\cdot\|_{W^m_q(G)}$ denote their norms, respectively. For a Banach space X with norm $\|\cdot\|_X$, X^d denotes the d-product space of X, while $\|\cdot\|_X$ denotes also the norm of X^d for the sake of simplicity. For a domain U in $\mathbb{C}$, $\mathbb{C}$ being the set of all complex number, $\mathrm{Anal}\,(U, X)$ denotes the set of all X-valued holomorphic functions defined on U. Σ_ϵ and $\Sigma_{\epsilon,\lambda_0}$ are sets for the resolvent parameter λ defined by

$$\Sigma_\epsilon = \{\lambda \in \mathbb{C} \setminus \{0\} \mid |\arg\lambda| \le \pi - \epsilon\}, \ \ \Sigma_{\epsilon,\lambda_0} = \{\lambda \in \Sigma_\epsilon \mid |\lambda| \ge \lambda_0\}.$$

The letter C denotes generic constants and $C_{a,b,c,\cdots}$ means that the constant $C_{a,b,c,\cdots}$ depends on a, b, c, $\cdots$. The values of constants C and $C_{a,b,c,\cdots}$ may change from line to line.

The following theorem is our main result in this paper.

Theorem 1.5. *Let $1 < q < \infty$ and $0 < \epsilon < \pi/2$. Assume that Ω is a uniform C^1 domain in $\mathbb{R}^N$ and the boundary of Ω consists of two C^1 hypersurfaces Γ_1 and Γ_2 with $\Gamma_1 \cap \Gamma_2 = \emptyset$. Let $X_q(\Omega)$ and $\mathcal{X}_q(\Omega)$ be functional spaces defined by*

$$X_q(\Omega) = \{(f, g, h_d, h_n) \mid f \in L_q(\Omega)^N, g \in L_q(\Omega), h_d, h_n \in W^1_q(\Omega)\},$$
$$\mathcal{X}_q(\Omega) = \{F = (F_1, \dots, F_6) \mid F_1, F_4, F_6 \in L_q(\Omega)^N, \ \ F_2, F_3, F_5 \in L_q(\Omega)\}.$$

Then, there exists a $\lambda_0 > 0$ and an operator family $\mathcal{A}(\lambda) \in \mathrm{Anal}\,(\Sigma_{\epsilon,\lambda_0}, \mathcal{L}(\mathcal{X}_q(\Omega), W^1_q(\Omega)))$ such that for any $\lambda \in \Sigma_{\epsilon,\lambda_0}$ and $(f, g, h_d, h_n) \in X_q(\Omega)$ $u = \mathcal{A}(\lambda)F_\lambda(f, g, h_d, h_n)$ is a unique solution to (1.3), where we have set $F_\lambda(f, g, h_d, h_n) = (f, \lambda^{-1/2}, \lambda^{1/2}h_d, \nabla h_d, h_n, \lambda^{-1/2}\nabla h_n)$.

Moreover, there exists a constant κ such that

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega), L_q(\Omega)^{N+1})}(\{(\lambda\frac{d}{d\lambda})^\ell(\lambda^{1/2}, \nabla)\mathcal{A}(\lambda) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \le \kappa \quad (\ell = 0, 1).$$

Finally, we discuss the generation of analytic semigroup and maximal L_p-L_q regularity results related to (1.3) as an application of Theorem 1.5. Let $W^{-1}_{q,\Gamma_1}(\Omega)$ be the dual space of $W^1_{q',\Gamma_1}(\Omega)$. It follow from the Hahn-Banach theorem that for any $F \in W^{-1}_{q,\Gamma_1}(\Omega)$ there exist $f \in L_q(\Omega)^N$ and $g \in L_q(\Omega)$ such that

$$F(\varphi) = -(f, \nabla\varphi)_\Omega + (g, \varphi)_\Omega \ \text{ for any } \varphi \in W^1_{q',\Gamma_1}(\Omega). \tag{1.5}$$

Let A be an operator defined by

$$Au(\varphi) = (\nabla u, \nabla\varphi)_\Omega \quad \text{for any } u \in W^1_{q,\Gamma_1}(\Omega) \text{ and } \varphi \in W^1_{q',\Gamma_1}(\Omega).$$

It follows from (1.3) and (1.5) that the resolvent problem: $\lambda u - Au = F$ is represented by

$$(\lambda u, \varphi)_\Omega + (\nabla u, \nabla\varphi)_\Omega = -(f, \nabla\varphi)_\Omega + (g, \varphi)_\Omega \text{ for any } \varphi \in W^1_{q',\Gamma_1}(\Omega) \tag{1.6}$$

subject to $u = 0$ on Γ_1. Since $\mathcal{R}$-boundedness implies boundedness, by Theorem 1.5 we see that the equation (1.6) admits a unique solution $\in W^1_{q,\Gamma_1}(\Omega)$ satisfying the estimate:

$$\|(\lambda^{1/2}u, \nabla u)\|_{L_q(\Omega)} \leq \kappa\|F\|_{W^{-1}_{q,\Gamma_1}(\Omega)} \tag{1.7}$$

for any $\lambda \in \Sigma_{\epsilon,\lambda_0}$ and $F \in W^{-1}_{q,\Gamma_1}(\Omega)$. Here, we may assume that $\lambda_0 \geq 1$. In addition, by (1.6) we have

$$\begin{aligned}|(\lambda u, \varphi)_\Omega| &\leq \|\nabla u\|_{L_q(\Omega)}\|\nabla\varphi\|_{L_{q'}(\Omega)} + \|f\|_{L_q(\Omega)}\|\nabla\varphi\|_{L_{q'}(\Omega)} + \|g\|_{L_q(\Omega)}\|\varphi\|_{L_{q'}(\Omega)} \\ &\leq C\kappa\|F\|_{W^{-1}_{q,\Gamma_1}(\Omega)}\|\varphi\|_{W^1_{q',\Gamma_1}(\Omega)},\end{aligned}$$

which furnishes that

$$\|\lambda u\|_{W^{-1}_{q,\Gamma_1}(\Omega)} \leq C\kappa\|F\|_{W^{-1}_{q,\Gamma_1}(\Omega)}. \tag{1.8}$$

Therefore, A generates an analytic semigroup $\{T(t)\}_{t\geq 0}$ on $W^{-1}_{q,\Gamma_1}(\Omega)$ satisfying the estimate:

$$\|T(t)F\|_{W^{-1}_{q,\Gamma_1}(\Omega)} + \|(t^{1/2}T(t)F, tT(t)F)\|_{L_q(\Omega)} \leq Ce^{\lambda_0 t}\|F\|_{W^{-1}_{q,\Gamma_1}(\Omega)} \tag{1.9}$$

for any $t > 0$ with some constant C.

Next, we consider the evolution equation:

$$u_t - Au = F \quad \text{in } \Omega, \quad u|_{\Gamma_1} = h_d|_{\Gamma_1} \tag{1.10}$$

for any $t \in \mathbb{R}$. Applying the Laplace transform to (1.10), we have

$$(\lambda\hat{u}, \varphi)_\Omega + (\nabla\hat{u}, \nabla\varphi)_\Omega = -(\hat{f}, \nabla\varphi)_\Omega + (\hat{g}, \varphi)_\Omega \quad \text{for any } \varphi \in W^1_{q',\Gamma_1}(\Omega) \tag{1.11}$$

subject to $\hat{u} = \hat{h}_d$ on Γ_1. Using the operator $\mathcal{A}(\lambda)$ given in Theorem 1.5, $\hat{u}$ is represented by

$$\hat{u} = \mathcal{A}(\lambda)(\hat{f}, \lambda^{-1/2}\hat{g}, \lambda^{1/2}\hat{h}_d, \nabla\hat{h}_d, 0, 0)$$

with $\lambda = \gamma + i\tau \in \mathbb{C}$. Let $\mathcal{L}^{-1}$ be the inverse Laplace transform, and then a unique solution u to (1.10) is represented by

$$u(t) = \mathcal{L}^{-1}[\mathcal{A}(\lambda)(\hat{f}, \lambda^{-1/2}\hat{g}, \lambda^{1/2}\hat{h}_d, \nabla\hat{h}_d, 0, 0)](t).$$

Therefore, by Theorem 1.2 we have

$$\begin{aligned}&\|e^{-\gamma t}u_t\|_{L_p(\mathbb{R}, W^{-1}_{q,\Gamma_1}(\Omega))} + \|e^{-\gamma t}(\Lambda^{1/2}_\gamma u, \nabla u)\|_{L_p(\mathbb{R}, L_q(\Omega))} \\ &\quad\leq C\kappa\|e^{-\gamma t}(f, \Lambda^{-1/2}_\gamma g, \Lambda^{1/2}_\gamma h_d, \nabla h_d)\|_{L_p(\mathbb{R}, L_q(\Omega))}\end{aligned}$$

for any $\gamma \geq \lambda_0$. Namely, the operator A has maximal L_p-L_q regularity. Here, we have set

$$\begin{aligned}\|e^{-\gamma t}v\|_{L_p(\mathbb{R},X)} &= \Big(\int_{-\infty}^{\infty}(e^{-\gamma t}\|v(t)\|_X)^p\,dt\Big)^{1/p}, \\ \|e^{-\gamma t}\Lambda^s_\gamma v\|_{L_p(\mathbb{R},X)} &= \Big(\int_{-\infty}^{\infty}(e^{-\gamma t}\|\mathcal{L}^{-1}[\lambda^s\hat{v}(\lambda)](t)\|_X)^p\,dt\Big)^{1/p}.\end{aligned}$$

2 Model Problems

2.1 A Model Problem in the whole space $\mathbb{R}^N$

Let us consider the problem:

$$\lambda(u,\varphi)_{\mathbb{R}^N} + (\nabla u, \nabla\varphi)_{\mathbb{R}^N} = -(f,\nabla\varphi)_{\mathbb{R}^N} + (g,\varphi)_{\mathbb{R}^N} \quad \text{for any } \varphi \in W_q^1(\mathbb{R}^N). \tag{2.1}$$

Instead of (2.1), we consider the equation: $(\lambda - \Delta)u = \operatorname{div} f + g$ and then using the Fourier transform and its inversion formula, we have

$$u(x) = \mathcal{F}_\xi^{-1}\Big[\frac{\mathcal{F}[\operatorname{div} f + g](\xi)}{\lambda + |\xi|^2}\Big](x) = \sum_{j=1}^N \mathcal{F}_\xi^{-1}\Big[\frac{i\xi_j \hat f_j(\xi)}{\lambda+|\xi|^2}\Big](x) + \mathcal{F}_\xi^{-1}\Big[\frac{\hat g(\xi)}{\lambda+|\xi|^2}\Big](x) \tag{2.2}$$

Here and hereafter, $\mathcal{F}[f](\xi) = \hat f(\xi)$ and $\mathcal{F}_\xi^{-1}[h(\xi)](x)$ denote the Fourier transform of $f(x)$ and the Fourier inverse transform of $h(\xi)$, respectively, which are defined exactly by

$$\mathcal{F}[f](\xi) = \hat f(\xi) = \int_{\mathbb{R}^N} e^{-ix\cdot\xi} f(x)\,dx, \quad \mathcal{F}_\xi^{-1}[h(\xi)](x) = \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} e^{ix\cdot\xi} h(\xi)\,d\xi.$$

To prove the R boundedness of the operators defined by the Fourier transform in $\mathbb{R}^N$, we use the following lemma due to Enomoto-Shibata [4, Theorem 3.3].

Theorem 2.1. *Let $1 < q < \infty$ and let Λ be a set in $\mathbb{C}$. Let $m(\lambda,\xi)$ be a function defined on $\Lambda \times (\mathbb{R}^N \setminus \{0\})$ such that for any multi-index $\alpha \in \mathbb{N}_0^N$ ($\mathbb{N}_0 = \mathbb{N}\cup\{0\}$) there exists a constant C_α depending on α and Λ such that*

$$|\partial_\xi^\alpha m(\lambda,\xi)| \le C_\alpha |\xi|^{-|\alpha|} \tag{2.3}$$

for any $(\lambda,\xi) \in \Lambda \times (\mathbb{R}^N \setminus \{0\})$. Let K_λ be an operator defined by $K_\lambda f = \mathcal{F}_\xi^{-1}[m(\lambda,\xi)\hat f(\xi)]$. Then, the set $\{K_\lambda \mid \lambda \in \Lambda\}$ is $\mathcal{R}$-bounded on $\mathcal{L}(L_q(\mathbb{R}^n))$ and

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N))}(\{K_\lambda \mid \lambda\in\Lambda\}) \le C_{q,N} \max_{|\alpha|\le N+2} C_\alpha \tag{2.4}$$

with some constant $C_{q,N}$ that depends solely on q and N.

Since $|\lambda + |\xi|^2| \ge 2\sin^2\frac{\epsilon}{2}(|\lambda| + |\xi|^2)$ for any $\lambda \in \Sigma_\epsilon$ and $\xi \in \mathbb{R}^N$, we see easily that $(\lambda + |\xi|^2)^{-1}$ satisfies the following multiplier conditions:

$$\begin{aligned} |\partial_\xi^\alpha[\lambda(\lambda+|\xi|^2)^{-1}]| &\le C_{\alpha,\epsilon}(|\lambda|^{1/2} + |\xi|)^{-|\alpha|}, \\ |\partial_\xi^\alpha[(\lambda^{1/2} i\xi_j)(\lambda+|\xi|^2)^{-1}]| &\le C_{\alpha,\epsilon}(|\lambda|^{1/2} + |\xi|)^{-|\alpha|}, \\ \partial_\xi^\alpha[(i\xi_j\xi_k)(\lambda+|\xi|^2)^{-1}]| &\le C_{\alpha,\epsilon}(|\lambda|^{1/2} + |\xi|)^{-|\alpha|}, \end{aligned} \tag{2.5}$$

for $j,k = 1,\ldots,N$ and any $\lambda \in \Sigma_\epsilon$ and $\xi \in \mathbb{R}^N$. Since

$$\lambda^{\frac12} u(x) = \sum_{j=1}^N \mathcal{F}_\xi^{-1}\Big[\frac{(\lambda^{1/2}) i\xi_j \hat f_j(\xi)}{\lambda+|\xi|^2}\Big](x) + \mathcal{F}_\xi^{-1}\Big[\frac{\lambda \mathcal{F}[\lambda^{-\frac12} g](\xi)}{\lambda+|\xi|^2}\Big](x),$$

$$\frac{\partial u}{\partial x_k}(x) = \sum_{j=1}^N \mathcal{F}_\xi^{-1}\Big[\frac{(i\xi_j\xi_k)\hat f_j(\xi)}{\lambda+|\xi|^2}\Big](x) + \mathcal{F}_\xi^{-1}\Big[\frac{(\lambda^{\frac12} i\xi_k)\mathcal{F}[\lambda^{-\frac12} g](\xi)}{\lambda+|\xi|^2}\Big](x)$$

Therefore, if we define an oprator $U_0(\lambda)$ by

$$U_0(\lambda)(F_1,F_2) = \sum_{j=1}^{N} \mathcal{F}_\xi^{-1}[i\xi_j \hat{F}_{1j}(\xi)(\lambda+|\xi|^2)^{-1}](x) + \mathcal{F}_\xi^{-1}[\lambda^{1/2}\hat{F}_2(\xi)(\lambda+|\xi|^2)^{-1}](x) \tag{2.6}$$

with $F = (F_{11}, \dots, F_{1N})$, then we have the following theorem.

Theorem 2.2. *Let $1 < q < \infty$ and $0 < \epsilon < \pi/2$. For any domain G in $\mathbb{R}^N$, we set*

$$X_{q0}(G) = \{(f,g) \mid f \in L_q(G)^N,\ \ g \in L_q(G)\},$$
$$\mathcal{X}_{q0}(G) = \{(F_1,F_2) \mid F_1 = (F_{11},\dots,F_{1N}) \in L_q(\mathbb{R}^N),\ \ F_2 \in L_q(\mathbb{R}^N)\}.$$

Let $U_0(\lambda)$ be the operator defined by (2.6). Then, $U_0(\lambda) \in \mathrm{Anal}\,(\Sigma_\epsilon, \mathcal{L}(\mathcal{X}_{q0}(\mathbb{R}^N), W_q^1(\mathbb{R}^N)))$, for any $\lambda \in \Sigma_\epsilon$ and $(f,g) \in X_q(\mathbb{R}^N)^N$ $u(x) = U_0(\lambda)(f, \lambda^{-1/2}g)$ is a unique solution to (2.1), and

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_0(\mathbb{R}^N),\ L_q(\mathbb{R}^N)^{N+1})}(\{(\lambda\frac{d}{d\lambda})^\ell U_0(\lambda) \mid \lambda \in \Sigma_\epsilon\}) \le \gamma_0 \tag{2.7}$$

with some constant γ_0 depending solely on ϵ, q and N.

2.2 A model problem in the half space $\mathbb{R}^N_+$, Dirichlet condition case.

In this subsection we consider the weak Dirichlet problem in the half-space $\mathbb{R}^N_+$:

$$(\lambda u, \varphi)_{\mathbb{R}^N_+} + (\nabla u, \nabla\varphi)_{\mathbb{R}^N_+} = -(f, \nabla\varphi)_{\mathbb{R}^N_+} + (g,\varphi)_{\mathbb{R}^N_+} \quad \text{for any } \varphi \in W^1_{q',0}(\mathbb{R}^N_+) \tag{2.8}$$

subject to $u = h_d$ on $\mathbb{R}^N_0$, where $W^1_{q,0}(G) = \{u \in W^1_q(G) \mid u|_{\partial G} = 0\}$, ∂G being the boundary of G, $\mathbb{R}^N_+ = \{x = (x_1,\dots,x_N) \in \mathbb{R}^N \mid x_N > 0\}$, and $\mathbb{R}^N_0 = \{x = (x_1,\dots,x_N) \in \mathbb{R}^N \mid x_N = 0\}$. Since $C_0^\infty(\mathbb{R}^N_+)$ is dense in $L_q(\mathbb{R}^N_+)$, we may assume that $f \in C_0^\infty(\mathbb{R}^N_+)^N$ and $g \in C_0^\infty(\mathbb{R}^N_+)$, and we consider the strong equation: $(\lambda - \Delta)u = \mathrm{div}\, f + g$ in $\mathbb{R}^N_+$ subject to $u = h_d$ on $\mathbb{R}^N_0$ instead of (2.8). Given function h defined on $\mathbb{R}^N_+$, h^e and h^o denote the even extension of h and the odd extension of h to $x_N < 0$, respectively. A unique solution $u(x)$ is given by

$$u(x) = \mathcal{F}_\xi^{-1}\Big[\frac{\mathcal{F}[(\mathrm{div}\, f)^o](\xi)}{\lambda+|\xi|^2}\Big](x) + \mathcal{F}_\xi^{-1}\Big[\frac{\mathcal{F}[g^o](\xi)}{\lambda+|\xi|^2}\Big](x) + \mathcal{F}_{\xi'}^{-1}[e^{-\omega_\lambda(\xi')x_N}\mathcal{F}_{\xi'}[h_d](\xi',0)](x') \tag{2.9}$$

with $\omega_\lambda(\xi') = \sqrt{\lambda + |\xi'|^2}$. Here, $\mathcal{F}_{\xi'}$ and $\mathcal{F}_{\xi'}^{-1}$ denote the partial Fourier transform and partial inverse fourier transform defined by

$$\mathcal{F}_{\xi'}[h_d](\xi', y_N) = \int_{\mathbb{R}^{N-1}} e^{-ix'\cdot\xi'} h_d(x', y_N)\, dx', \quad \mathcal{F}_{\xi'}^{-1}[g(\xi')](x') = \frac{1}{(2\pi)^{N-1}} \int_{\mathbb{R}^{N-1}} e^{ix'\cdot\xi'} g(\xi')\, d\xi'$$

with $x' = (X_1, \dots, x_{N-1})$ and $\xi' = (\xi_1, \dots, \xi_{N-1})$. To obtain

$$\begin{aligned}
u(x) &= \sum_{j=1}^{N-1} \mathcal{F}_\xi^{-1}\Big[\frac{i\xi_j\mathcal{F}[f_j^o](\xi)}{\lambda+|\xi|^2}\Big](x) + \mathcal{F}_\xi^{-1}\Big[\frac{i\xi_N\mathcal{F}[f_N^e](\xi)}{\lambda+|\xi|^2}\Big](x) \\
&+ \mathcal{F}_\xi^{-1}\Big[\frac{\lambda^{1/2}\mathcal{F}[\lambda^{-1/2}g^o](\xi)}{\lambda+|\xi|^2}\Big](x) + \int_0^\infty \mathcal{F}_{\xi'}^{-1}[e^{-\omega_\lambda(\xi')(x_N+y_N)}\mathcal{F}_{\xi'}[D_N h_d](\xi', y_N)](x')\, dy_N \\
&+ \sum_{j=1}^{N-1} \int_0^\infty \mathcal{F}_{\xi'}^{-1}[e^{-\omega_\lambda(\xi')(x_N+y_N)} \frac{i\xi_j}{\omega_\lambda(\xi')}\mathcal{F}_{\xi'}[D_j h_d](\xi', y_N)](x')\, dy_N \\
&- \int_0^\infty e^{-\omega_\lambda(\xi')(x_N+y_N)} \frac{\lambda^{1/2}}{\omega_\lambda(\xi')}\mathcal{F}_{\xi'}[\lambda^{1/2}h_d](\xi', y_N)](x')\, dy_N,
\end{aligned} \tag{2.10}$$

we use the formula: $(\operatorname{div} f)^o = \sum_{j=1}^{N-1} D_j(f_j^o)+D_N(f_N^e)$ with $D_j = \partial/\partial x_j$, $\omega_\lambda(\xi') = \lambda\omega_\lambda(\xi')^{-1} - \sum_{j=1}^{N-1}(i\xi_j)(i\xi_j)\omega_\lambda(\xi')^{-1}$ and the Volevich trick:

$$e^{-\omega_\lambda(\xi')x_N}\mathcal{F}_{\xi'}^{-1}[h_d](\xi',0) = -\int_0^\infty \frac{\partial}{\partial y_N}[e^{-\omega_\lambda(\xi')(x_N+y_N)}\mathcal{F}_{\xi'}^{-1}[h_d](\xi',y_N)]\,dy_N.$$

In view of (2.10), we define an operator $\mathcal{S}_d(\lambda)$ by

$$\begin{aligned}
\mathcal{S}_d(\lambda)(F_1,F_2,F_3,F_4) &= \sum_{j=1}^{N-1}\mathcal{F}_\xi^{-1}\Big[\frac{i\xi_j\mathcal{F}[F_{1j}^o](\xi)}{\lambda+|\xi|^2}\Big](x) + \mathcal{F}_\xi^{-1}\Big[\frac{i\xi_N\mathcal{F}[F_{1N}^e](\xi)}{\lambda+|\xi|^2}\Big](x)\\
&+\mathcal{F}_\xi^{-1}\Big[\frac{\lambda^{1/2}\mathcal{F}[F_2^o](\xi)}{\lambda+|\xi|^2}\Big](x) + \int_0^\infty \mathcal{F}_{\xi'}^{-1}[e^{-\omega_\lambda(\xi')(x_N+y_N)}\mathcal{F}_{\xi'}[F_{4N}](\xi',y_N)](x')\,dy_N\\
&+\sum_{j=1}^{N-1}\int_0^\infty \mathcal{F}_{\xi'}^{-1}[e^{-\omega_\lambda(\xi')(x_N+y_N)}\frac{i\xi_j}{\omega_\lambda(\xi')}\mathcal{F}_{\xi'}[F_{4j}](\xi',y_N)](x')\,dy_N\\
&-\int_0^\infty e^{-\omega_\lambda(\xi')(x_N+y_N)}\frac{\lambda^{1/2}}{\omega_\lambda(\xi')}\mathcal{F}_{\xi'}[F_3](\xi',y_N)](x')\,dy_N
\end{aligned} \tag{2.11}$$

with F_2, $F_3 \in L_q(\mathbb{R}_+^N)$ and F_1, $F_4 = (F_{41},\dots,F_{4,N}) \in L_q(\mathbb{R}_+^N)^N$. Combining (2.10) and (2.11), we have

$$u(x) = \mathcal{S}_d(\lambda)F_\lambda^d(f,g,h_d). \tag{2.12}$$

with $F_\lambda^d(f,g,h_d) = (f,\lambda^{-1/2}g,\lambda^{1/2}h_d,\nabla h_d)$. To prove the $\mathcal{R}$-boundedness of $\mathcal{S}_d(\lambda)$, we use the following lemma due to Shibata and Shimizu [9, Lemma 5.4]

Lemma 2.3. *Let $0<\epsilon<\pi/2$ and $1<q<\infty$. Let m_1 and m_2 be functions defined on $\Sigma_\epsilon\times\mathbb{R}^{N-1}\setminus\{0\}$ that satisfy the multiplier conditions:*

$$\begin{aligned}
|\partial_{\xi'}^{\alpha'}[(\lambda\frac{d}{d\lambda})^\ell m_1(\lambda,\xi')]| &\le C_{\alpha'}(|\lambda|^{1/2}+|\xi'|)^{-|\alpha'|} \quad (\ell=0,1),\\
|\partial_{\xi'}^{\alpha'}[(\lambda\frac{d}{d\lambda})^\ell m_2(\lambda,\xi')]| &\le C_{\alpha'}|\xi'|^{-|\alpha'|} \quad (\ell=0,1)
\end{aligned} \tag{2.13}$$

for any $\alpha'=(\alpha_1,\dots,\alpha_{N-1})\in\mathbb{N}_0^{N-1}$† and $(\lambda,\xi')\in\Sigma_\epsilon\times\mathbb{R}^{N-1}\setminus\{0\}$, where $\partial_{\xi'}^{\alpha'}=\partial_{\xi_1}^{\alpha_1}\cdots\partial_{\xi_{N-1}}^{\alpha_{N-1}}$. Let $K_j(\lambda)$ $(j=1,2)$ be operators defined by

$$\begin{aligned}
[K_1(\lambda)g](x) &= \int_0^\infty \mathcal{F}_{\xi'}^{-1}[m_1(\lambda,\xi')\lambda^{1/2}e^{-\omega_\lambda(\xi')(x_N+y_N)}\mathcal{F}_{\xi'}[g](\xi',y_N)](x')\,dy_N,\\
[K_2(\lambda)g](x) &= \int_0^\infty \mathcal{F}_{\xi'}^{-1}[m_2(\lambda,\xi')|\xi'|e^{-\omega_\lambda(\xi')(x_N+y_N)}\mathcal{F}_{\xi'}[g](\xi',y_N)](x')\,dy_N.
\end{aligned}$$

Then, there exists a constant β_0 depending on ϵ, q and N such that

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}_+^N))}(\{(\lambda\frac{d}{d\lambda})^\ell K_j(\lambda)\mid\lambda\in\Sigma_\epsilon\})\le\beta_0 \quad (\ell=0,1,\ j=1,2).$$

Since $\lambda^{1/2}/\omega_\lambda(\xi')$ and $i\xi_j/\omega_\lambda(\xi')$ satisfy the multiplier conditions (2.13), respectively, by Lemma 2.3 and Theorem 3.4, we have the following theorem.

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†$\mathbb{N}$ denotes the set of all natural numbers and $\mathbb{N}_0=\mathbb{N}\cup\{0\}$.

Theorem 2.4. *Let $1 < q < \infty$ and $0 < \epsilon < \pi/2$. For any domain G in $\mathbb{R}^N$, we set*

$$X_{qd}(G) = \{(f, g, h_d) \mid f \in L_q(G)^N, \;\; g \in L_q(G), \;\; h_d \in W_q^1(G)\},$$
$$\mathcal{X}_{qd}(G) = \{(F_1, F_2, F_3, F_4) \mid F_1, F_4 \in L_q(G)^N, \;\; F_2, F_3 \in L_q(G)\}.$$

Let $\mathcal{S}_d(\lambda)$ be the operator defined in (2.11). Then, $\mathcal{S}_d(\lambda) \in \mathrm{Anal}\,(\Sigma_\epsilon, \mathcal{L}(X_{qd}(\mathbb{R}^N_+), W_q^1(\mathbb{R}^N_+)))$, for any $\lambda \in \Sigma_\epsilon$ and $(f, g, h_d) \in X_{qd}(\mathbb{R}^N_+)$ $u = \mathcal{S}_d(\lambda)F^d_\lambda(f, g, h_d)$ is a unique solution to (2.8), and

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_{qd}(\mathbb{R}^N_+), L_q(\mathbb{R}^N_+)^{N+1})}(\{(\lambda\frac{d}{d\lambda})^\ell(\lambda^{1/2}, \nabla)\mathcal{S}_d(\lambda) \mid \lambda \in \Sigma_\epsilon\}) \leq \beta_0 \quad (\ell = 0, 1)$$

with some constant β_0 depending on ϵ, q and N.

2.3 A model problem in the half space $\mathbb{R}^N_+$, Neumann condition case.

In this subsection we consider the weak Neumann problem in the half-space $\mathbb{R}^N_+$:

$$(\lambda u, \varphi)_{\mathbb{R}^N_+} + (\nabla u, \nabla\varphi)_{\mathbb{R}^N_+} = -(f, \nabla\varphi)_{\mathbb{R}^N_+} + (g, \varphi)_{\mathbb{R}^N_+} + < h_n, \varphi >_{\mathbb{R}^N_0} \tag{2.14}$$

for any $\varphi \in W^1_{q'}(\mathbb{R}^N_+)$, where $< a, b >_{\mathbb{R}^N_0} = \int_{\mathbb{R}^{N-1}} a(x')b(x')\,dx'$. We consider the strong equation: $(\lambda - \Delta)u = \operatorname{div} f + g$ in $\mathbb{R}^N_+$ subject to $D_N u = h_n$ on $\mathbb{R}^N_0$ instead of (2.14). Then, its unique solution is given by

$$u(x) = \mathcal{F}^{-1}_\xi\Big[\frac{\mathcal{F}[(\operatorname{div} f)^e](\xi)}{\lambda + |\xi|^2}\Big](x) + \mathcal{F}^{-1}_\xi\Big[\frac{\mathcal{F}[g^e](\xi)}{\lambda + |\xi|^2}\Big](x) + \mathcal{F}^{-1}_{\xi'}\Big[\frac{e^{-\omega_\lambda(\xi')x_N}}{\omega_\lambda(\xi')}\mathcal{F}_{\xi'}[h_n](\xi', 0)\Big](x'). \tag{2.15}$$

Since we may assume that $f \in C_0^\infty(\mathbb{R}^N_+)^N$, we have $(\operatorname{div} f)^e = \sum_{j=1}^{N-1} D_j(f_j^e) + D_N(f_N^o)$, so that

$$\begin{aligned} u(x) = &\sum_{j=1}^{N-1}\mathcal{F}^{-1}_\xi\Big[\frac{i\xi_j\mathcal{F}[f_j^e](\xi)}{\lambda + |\xi|^2}\Big](x) + \mathcal{F}^{-1}_\xi\Big[\frac{i\xi_N\mathcal{F}[f_N^o](\xi)}{\lambda + |\xi|^2}\Big](x) \\ &+ \mathcal{F}^{-1}_\xi\Big[\frac{\lambda^{1/2}\mathcal{F}[\lambda^{-1/2}g^e](\xi)}{\lambda + |\xi|^2}\Big](x) \\ &+ \int_0^\infty \mathcal{F}^{-1}_{\xi'}[e^{-\omega_\lambda(\xi')(x_N+y_N)}\frac{\lambda^{1/2}}{\omega_\lambda(\xi')}\mathcal{F}_{\xi'}[\lambda^{-1/2}D_N h_n](\xi', y_N)](x')\,dy_N \\ &- \int_0^\infty e^{-\omega_\lambda(\xi')(x_N+y_N)}\mathcal{F}_{\xi'}[h_n](\xi', y_N)](x')\,dy_N. \end{aligned} \tag{2.16}$$

In view of (2.16), we define an operator $\mathcal{S}_n(\lambda)$ by

$$\begin{aligned} \mathcal{S}_n(\lambda)(F_1, F_2, F_5, F_6) = &\sum_{j=1}^{N-1}\mathcal{F}^{-1}_\xi\Big[\frac{i\xi_j\mathcal{F}[F_{1j}^e](\xi)}{\lambda + |\xi|^2}\Big](x) + \mathcal{F}^{-1}_\xi\Big[\frac{i\xi_N\mathcal{F}[F_{1N}^o](\xi)}{\lambda + |\xi|^2}\Big](x) \\ + \mathcal{F}^{-1}_\xi\Big[\frac{\lambda^{1/2}\mathcal{F}[F_2^e](\xi)}{\lambda + |\xi|^2}\Big](x) &+ \int_0^\infty \mathcal{F}^{-1}_{\xi'}[e^{-\omega_\lambda(\xi')(x_N+y_N)}\frac{\lambda^{1/2}}{\omega_\lambda(\xi')}\mathcal{F}_{\xi'}[F_{6N}](\xi', y_N)](x')\,dy_N \\ - \int_0^\infty e^{-\omega_\lambda(\xi')(x_N+y_N)}&\mathcal{F}_{\xi'}[F_5](\xi', y_N)](x')\,dy_N \end{aligned} \tag{2.17}$$

with F_2, $F_5 \in L_q(\mathbb{R}^N_+)$ and F_1, $F_6 = (F_{61}, \ldots, F_{6N}) \in L_q(\mathbb{R}^N_+)^N$. Combining (2.16) and (2.17), we have

$$u(x) = \mathcal{S}_n(\lambda)F^n_\lambda(f, g, h_n) \tag{2.18}$$

with $F^n_\lambda(f, g, h_n) = (f, \lambda^{-1/2}g, h_n, \lambda^{-1/2}\nabla h_n)$. Applying Lemma 2.3 and Theorem 3.4, we have the following theorem.

Theorem 2.5. *Let $1 < q < \infty$ and $0 < \epsilon < \pi/2$. For any domain G in $\mathbb{R}^N$, we set*

$$X_{qn}(G) = \{(f, g, h_n) \mid f \in L_q(G)^N, \;\; g \in L_q(G), \;\; h_n \in W^1_q(G)\},$$
$$\mathcal{X}_{qn}(G) = \{(F_1, F_2, F_5, F_6) \mid F_1, F_6 \in L_q(G)^N, \;\; F_2, F_5 \in L_q(G)\}.$$

Let $\mathcal{S}_n(\lambda)$ be the operator defined in (2.17). Then, $\mathcal{S}_n(\lambda) \in \mathrm{Anal}\,(\Sigma_\epsilon, \mathcal{L}(X_{qn}(\mathbb{R}^N_+), W^1_q(\mathbb{R}^N_+)))$, for any $\lambda \in \Sigma_\epsilon$ and $(f, g, h_n) \in X_{qn}(\mathbb{R}^N_+)$ $u = \mathcal{S}_n(\lambda)F^n_\lambda(f, g, h_n)$ is a unique solution to (2.14), and

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_{qn}(\mathbb{R}^N_+), L_q(\mathbb{R}^N_+)^{N+1})}(\{(\lambda\frac{d}{d\lambda})^\ell(\lambda^{1/2}, \nabla)\mathcal{S}_n(\lambda) \mid \lambda \in \Sigma_\epsilon\}) \leq \beta_0 \quad (\ell = 0, 1)$$

with some constant β_0 depending on ϵ, q and N.

3 $\mathcal{R}$-boundedness of solution operators in a bent-half space

Let $\Phi : \mathbb{R}^N \to \mathbb{R}^N$ be a bijection of C^1 class and let Φ^{-1} be its inverse map. We assume that $\nabla\Phi = \mathcal{A} + B(x)$ and $\nabla\Phi^{-1} = \mathcal{A}_{-1} + B_{-1}(x)$, where $\mathcal{A}$ and $\mathcal{A}_{-1}$ are orthonormal matrices with constant coefficients and $B(x)$ and $B_{-1}(x)$ are matrices of functions in $L_\infty(\mathbb{R}^N)$ such that

$$\|(B, B_{-1})\|_{L_\infty(\mathbb{R}^N)} \leq M_1. \tag{3.1}$$

We will choose M_1 small enough eventually, so that we may assume that $0 < M_1 \leq 1$ in the following. Set $\Omega_+ = \Phi(\mathbb{R}^N_+)$ and $\Gamma_+ = \Phi(\mathbb{R}^N_0)$. Let $\mathfrak{g}$ be a function defined by $\det(\nabla\Phi) = 1 + \mathfrak{g}$. We choose $0 < M_1 \leq 1$ so small that

$$\|\mathfrak{g}\|_{L_\infty(\mathbb{R}^N)} \leq C_N M_1 \tag{3.2}$$

with some constant depending solely on N. In this section, we consider the weak Dirichlet problem and the weak Neumann problem on Ω_+.

3.1 Dirichlet boundary condition case

In this subsection, we consider the variational problem:

$$(\lambda u, \varphi)_{\Omega_+} + (\nabla u, \nabla\varphi)_{\Omega_+} = -(f, \nabla\varphi)_{\Omega_+} + (g, \varphi)_{\Omega_+} \quad \text{for any } \varphi \in W^1_{q',0}(\Omega_+), \tag{3.3}$$

subject to $u = h_d$ on Γ_+. By the change of variable: $y = \Phi(x)$, we transform (3.3) into the half-space problem. Setting $u \circ \Phi(x) = v(x)$ and $\varphi \circ \Phi(x) = \psi(x)$ and using the formula: $\frac{\partial x_k}{\partial y_j} = \mathcal{A}_{jk} + B_{kj}(x)$ with $\mathcal{A}_{-1} = (\mathcal{A}_{kj})$ and $B_{-1}(x) = (B_{kj}(x))$, we have

$$\begin{aligned}(\nabla u, \nabla\varphi)_{\Omega_+} &= \sum_{j,k,\ell=1}^N \int_{\mathbb{R}^N_+} (\mathcal{A}_{kj} + B_{kj}(x))(\mathcal{A}_{\ell j} + B_{\ell j}(x))\frac{\partial v(x)}{\partial x_j}\frac{\partial\psi(x)}{\partial x_\ell}(1 + \mathfrak{g}(x))\,dx \\ &= (\nabla v, \nabla\psi)_{\mathbb{R}^N_+} + (\mathcal{P}\nabla v, \nabla\psi)_{\mathbb{R}^N_+},\end{aligned}$$

with

$$(\mathcal{P}\nabla v, \nabla\psi)_{\mathbb{R}^N_+} = (\mathfrak{g}\nabla v, \nabla\psi)_{\mathbb{R}^N_+} + ((1 + \mathfrak{g})\{\sum_{j=1}^N(\mathcal{A}_{kj}B_{\ell j} + B_{kj}\mathcal{A}_{\ell j} + B_{kj}B_{\ell j})\}\frac{\partial v}{\partial x_k}, \frac{\partial\psi}{\partial x_\ell})_{\mathbb{R}^N_+}.$$

In the similar way, we have

$$(f, \nabla\varphi)_{\Omega_+} = \sum_{j,k=1}^{N} ((1+\mathfrak{g})f_j \circ \Phi, (\mathcal{A}_{kj} + \mathcal{B}_{kj})\frac{\partial\psi}{\partial x_k})_{\mathbb{R}^N_+} = (F, \nabla\psi)_{\mathbb{R}^N_+},$$

where we have set $F = (F_1, \ldots, F_N)$ and $F_k = \sum_{j=1}^N (1+\mathfrak{g})(\mathcal{A}_{kj} + \mathcal{B}_{kj})f_j \circ \Phi$. Setting $G = (1+\mathfrak{g})g \circ \Phi$ and $H_d = h_d \circ \Phi$, finally we arrive at the variational equation:

$$(\lambda v, \psi)_{\mathbb{R}^N_+} + (\lambda\mathfrak{g}v, \psi)_{\mathbb{R}^N_+} + (\nabla v, \nabla\psi)_{\mathbb{R}^N_+} + (\mathcal{P}\nabla v, \nabla\psi)_{\mathbb{R}^N_+} = (F, \nabla\psi)_{\mathbb{R}^N_+} + (G, \psi)_{\mathbb{R}^N_+} \tag{3.4}$$

for any $\psi \in W^1_{q',0}(\mathbb{R}^N_+)$, subject to $v = H_d$ on $\mathbb{R}^N_0$. Let $\mathcal{S}_d(\lambda)$ be the operator given in Theorem 2.4. Inserting the formula: $v = \mathcal{S}_d(\lambda)F^d_\lambda(F, G, H_d)$ into (3.4), we have

$$\begin{aligned}
&(\lambda v, \psi)_{\mathbb{R}^N_+} + (\lambda\mathfrak{g}v, \psi)_{\mathbb{R}^N_+} + (\nabla v, \nabla\psi)_{\mathbb{R}^N_+} + (\mathcal{P}\nabla v, \nabla\psi)_{\mathbb{R}^N_+} \\
&\quad = -(F - \mathcal{P}\nabla\mathcal{S}_d(\lambda)F^d_\lambda(F, G, H_d), \nabla\psi)_{\mathbb{R}^N_+} + (G + \lambda\mathfrak{g}\mathcal{S}_d(\lambda)F^d_\lambda(F, G, H_d), \psi)_{\mathbb{R}^N_+}
\end{aligned} \tag{3.5}$$

for any $\psi \in W^1_{q',0}(\mathbb{R}^N_+)$, subject to $v = H_d$ on $\mathbb{R}^N_0$. Setting $\mathcal{F}_1(\lambda)F^d = -\mathcal{P}\nabla\mathcal{S}_d(\lambda)F^d$ and $\mathcal{F}_2(\lambda)F^d = \lambda\mathfrak{g}\mathcal{S}_d(\lambda)F^d$ with $F^d = (F_1, F_2, F_3, F_4)$, we write (3.5) as follows:

$$\begin{aligned}
&(\lambda(1+\mathfrak{g})v, \psi)_{\mathbb{R}^N_+} + ((I + \mathcal{P})\nabla v, \nabla\psi)_{\mathbb{R}^N_+} \\
&= -(F + \mathcal{F}_1(\lambda)F^d_\lambda(F, G, H_d), \nabla\psi)_{\mathbb{R}^N_+} + (G + \mathcal{F}_2(\lambda)F^d_\lambda(F, G, H_d), \psi)_{\mathbb{R}^N_+} \quad \text{for any } \psi \in W^1_{q',}(\mathbb{R}^N_+),
\end{aligned}$$

subject to $v = H_d$ on $\mathbb{R}^N_0$. Setting $\mathcal{F}(\lambda)F^d = (\mathcal{F}_1(\lambda)F^d, \mathcal{F}_2(\lambda)F^d, 0)$, we have

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\mathbb{R}^N_+))}(\{(\lambda\frac{d}{d\lambda})^\ell F^d_\lambda \mathcal{F}(\lambda) \mid \lambda \in \Sigma_\epsilon\}) \leq C_N M_1 \beta_0 \quad (\ell = 0, 1) \tag{3.6}$$

where β_0 is the same constant as in Theorem 2.4. To prove (3.6), we use the following lemmas.

Lemma 3.1. (1) *Let X and Y be Banach spaces, and let $\mathcal{T}$ and $\mathcal{S}$ be $\mathcal{R}$-bounded families in $\mathcal{L}(X, Y)$. Then, $\mathcal{T} + \mathcal{S} = \{T + S \mid T \in \mathcal{T},\ S \in \mathcal{S}\}$ is also an $\mathcal{R}$-bounded family in $\mathcal{L}(X, Y)$ and*

$$\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T} + \mathcal{S}) \leq \mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T}) + \mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{S}).$$

(2) *Let X, Y and Z be Banach spaces, and let $\mathcal{T}$ and $\mathcal{S}$ be $\mathcal{R}$-bounded families in $\mathcal{L}(X, Y)$ and $\mathcal{L}(Y, Z)$, respectively. Then, $\mathcal{S}\mathcal{T} = \{ST \mid T \in \mathcal{T},\ S \in \mathcal{S}\}$ is also an $\mathcal{R}$-bounded family in $\mathcal{L}(X, Z)$ and*

$$\mathcal{R}_{\mathcal{L}(X,Z)}(\mathcal{S}\mathcal{T}) \leq \mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T})\mathcal{R}_{\mathcal{L}(Y,Z)}(\mathcal{S}).$$

Lemma 3.2. *Let $1 < p,\ q < \infty$ and let D be a domain in $\mathbb{R}^N$.*
(1) *Let $m(\lambda)$ be a bounded function defined on a subset Λ in a complex plane $\mathbb{C}$ and let $M_m(\lambda)$ be a multiplication operator with $m(\lambda)$ defined by $M_m(\lambda)f = m(\lambda)f$ for any $f \in L_q(D)$. Then,*

$$\mathcal{R}_{\mathcal{L}(L_q(D))}(\{M_m(\lambda) \mid \lambda \in \Lambda\}) \leq C_{N,q,D}\|m\|_{L_\infty(\Lambda)}.$$

(2) *Let $n(\tau)$ be a C^1 funtion defined on $\mathbb{R} \setminus \{0\}$ that satisfies the conditions: $|n(\tau)| \leq \gamma$ and $|\tau n'(\tau)| \leq \gamma$ with some constant $\gamma > 0$ for any $\tau \in \mathbb{R} \setminus \{0\}$. Let T_n be an operator valued Fourier multiplier defined by $T_n f = \mathcal{F}^{-1}[n\mathcal{F}[f]]$ for any f with $\mathcal{F}[\phi] \in \mathcal{D}(\mathbb{R}, L_q(D))$. Then, T_n is extended to a bounded linear operator from $L_p(\mathbb{R}, L_q(D))$ into itself. Moreover, denoting this extension also by T_n, we have*

$$\|T_n\|_{\mathcal{L}(L_p(\mathbb{R}, L_q(D)))} \leq C_{p,q,D}\gamma.$$

Remark 3.3. For proofs of Lemma 3.1 and Lemma 3.2, we refer to [3, p.28, 3.4.Proposition and p.27, 3.2.Remarks (4)] (cf. also Bourgain [2]), respectively.

For any natural number n, $\{\lambda_\ell\}_{\ell=1}^n \subset \Sigma_\epsilon$, $\{F_\ell\}_{\ell=1}^n \subset \mathcal{X}_q(\mathbb{R}^N_+)$ and sequence $\{r_\ell(u)\}_{\ell-1}^n$ of independent, symmetric, $\{-1,1\}$-valued random variable on $[0,1]$, using Theorem 2.4 we have

$$\begin{aligned}
\int_0^1 \|\sum_{\ell=1}^n r_\ell(u)\mathcal{F}_1(\lambda_\ell)F_\ell\|^q_{L_q(\mathbb{R}^N_+)}\,du &\leq (C_N M_1)^q \int_0^1 \|\sum_{\ell=1}^n r_\ell(u)\nabla\mathcal{S}_d(\lambda_\ell)F_\ell\|^q_{L_q(\mathbb{R}^N_+)}\,du \\
&\leq (C_N M_1\beta_0)^q \int_0^1 \|\sum_{\ell=1}^n r_\ell(u)F_\ell\|^q_{L_q(\mathbb{R}^N_+)}\,du, \\
\int_0^1 \|\sum_{\ell=1}^n r_\ell(u)\lambda_\ell^{-1/2}\mathcal{F}_2(\lambda_\ell)F_\ell\|^q_{L_q(\mathbb{R}^N_+)}\,du &\leq (C_N M_1)^q \int_0^1 \|\sum_{\ell=1}^n r_\ell(u)\lambda_\ell^{1/2}\mathcal{S}_d(\lambda_\ell)F_\ell\|^q_{L_q(\mathbb{R}^N_+)}\,du \\
&\leq (C_N M_1\beta_0)^q \int_0^1 \|\sum_{\ell=1}^n r_\ell(u)F_\ell\|^q_{L_q(\mathbb{R}^N_+)}\,du,
\end{aligned}$$

Note that $\|F^d_\lambda(F,G,H_d)\|_{L_q(\mathbb{R}^N_+)} = \|(F,\lambda^{-1/2}G,\lambda^{1/2}H_d,\nabla H_d)\|_{L_q(\mathbb{R}^N_+)}$ give us equivalent norms on $X_q(\mathbb{R}^N_+)$ for $\lambda \neq 0$. Since

$$\|F^d_\lambda\mathcal{F}(\lambda)F^d_\lambda(F,G,H_d)\|_{L_q(\mathbb{R}^N_+)} \leq C_N M_1\beta_0\|F^d_\lambda(F,G,H_d)\|_{L_q(\mathbb{R}^N_+)}$$

as follows from (3.6) (cf. the definition of $\mathcal{R}$-boundeness with $\ell=1$ in Definition 1.1), choosing $0 < M_1 \leq 1$ so small that $C_N M_1\beta_0 \leq 1/2$, we see that $(I+F^d_\lambda)^{-1}\mathcal{F}(\lambda)$ exists in $\mathcal{L}(X_q(\mathbb{R}^N_+))$ for any $\lambda \in \Sigma_\epsilon$, and therefore $v = \mathcal{S}_d(\lambda)F^d_\lambda(I+\mathcal{F}(\lambda)F^d_\lambda)^{-1}(F,G,H_d)$ is a unique solution to (3.5). Moreover, we have

$$F^d_\lambda(I+\mathcal{F}(\lambda)F^d_\lambda)^{-1} = F^d_\lambda + \sum_{\ell=1}^\infty(-1)^\ell F^d_\lambda(\mathcal{F}(\lambda)F^d_\lambda)^\ell = (I+F^d_\lambda\mathcal{F}(\lambda))^{-1}F^d_\lambda,$$

which furnishes that $\mathcal{S}_d(\lambda)F^d_\lambda(I+\mathcal{F}(\lambda)F^d_\lambda)^{-1} = \mathcal{S}_d(\lambda)(I+F^d_\lambda\mathcal{F}(\lambda))^{-1}F^d_\lambda$. Setting $\mathcal{S}_{bd}(\lambda) = \mathcal{S}_d(\lambda)(I+F^d_\lambda\mathcal{F}(\lambda))^{-1}$, by (3.6) and Theorem 2.4 we have

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\mathbb{R}^N_+),L_q(\mathbb{R}^N_+)^{N+1})}(\{(\lambda\frac{d}{d\lambda})^\ell\mathcal{S}_{bd}(\lambda) \mid \lambda \in \Sigma_\epsilon\}) \leq 2\beta_0 \quad (\ell = 0,1)$$

and the solution v to (3.4) is represented by $v = \mathcal{S}_{bd}(\lambda)F^d_\lambda(F,G,H_d)$. By the change of variable: $x = \Phi^{-1}(y)$ we have the following theorem.

Theorem 3.4. *Let $1 < q < \infty$ and $0 < \epsilon < \pi/2$. Then, there exists a constant M_1 with $0 < M_1 \leq 1$ depending on q, N and ϵ such that if the condition (3.1) holds, then the following assertion holds: There exists an operator family $\mathcal{T}_d(\lambda) \in \mathrm{Anal}\,(\Sigma_\epsilon, \mathcal{L}(\mathcal{X}_{qd}(\Omega_+), W^1_q(\Omega_+)))$ such that $u = \mathcal{T}_d(\lambda)F^d_\lambda(f,g,h_d)$ is a unique solution to (3.3) for any $(f,g,h_d) \in X_{qd}(\Omega_+)$ and $\lambda \in \Sigma_\epsilon$, and*

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_{qd}(\Omega_+),L_q(\Omega_+)^{N+1})}(\{(\lambda\frac{d}{d\lambda})^\ell(\lambda^{1/2},\nabla)\mathcal{T}_d(\lambda) \mid \lambda \in \Sigma_\epsilon\}) \leq \beta_1$$

with some constant β_1 depending on β_0, q, ϵ and N.

3.2 Neumann boundary condition case

In this subsection, we consider the variational problem:

$$(\lambda u, \varphi)_{\Omega_+} + (\nabla u, \nabla\varphi)_{\Omega_+} = -(f, \nabla\varphi)_{\Omega_+} + (g, \varphi)_{\Omega_+} + < h_n, \varphi >_{\Gamma_+} \tag{3.7}$$

for any $\varphi \in W^1_{q'}(\Omega_+)$, where $< h_n, \varphi >_{\Gamma_+} = \int_{\Gamma_+} h_n \varphi \, dS$, dS being the surface element of Γ_+. Employing the same argument as in Subsec. 3.1, we transfer (3.7) to the half-space problem:

$$(\lambda(1+\mathfrak{g})v, \psi)_{\mathbb{R}^N_+} + ((I+\mathcal{P})\nabla v, \nabla\varphi)_{\mathbb{R}^N_+} = -(F, \nabla\psi)_{\mathbb{R}^N_+} + (G, \psi)_{\mathbb{R}^N_+} + < H_n, \psi >_{\mathbb{R}^N_0} \tag{3.8}$$

for any $\psi \in W^1_{q'}(\mathbb{R}^N_+)$. Let $\mathcal{S}_n(\lambda)$ be the operator given in Theorem 2.5. Inserting the formula: $v = \mathcal{S}_n(\lambda) F^n_\lambda(F, G, H_n)$ into (3.8), we have

$$\begin{aligned}(\lambda(1+\mathfrak{g})v, \psi)_{\mathbb{R}^N_+} + ((I+\mathcal{P})\nabla v, \nabla\psi)_{\mathbb{R}^N_+} &= -(F - \mathcal{P}\nabla\mathcal{S}_n(\lambda)F^n_\lambda(F, G, H_n), \nabla\psi)_{\mathbb{R}^N_+} \\ &+ (G + \lambda\mathfrak{g}\mathcal{S}_n(\lambda)F^n_\lambda(F, G, H_n), \psi)_{\mathbb{R}^N_+} + < H_n, \psi >_{\mathbb{R}^N_0} \quad \text{for any } \psi \in W^1_{q'}(\mathbb{R}^N_+).\end{aligned} \tag{3.9}$$

Setting $\mathcal{F}(\lambda)F^n = (\mathcal{P}\nabla\mathcal{S}_n(\lambda)F^n, \lambda\mathfrak{g}\mathcal{S}_n(\lambda)F^n, 0)$ with $F^n = (F_1, F_2, F_5, F_6)$, by Theorem 2.5 we have

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\mathbb{R}^N_+))}(\{(\lambda\frac{d}{d\lambda})^\ell F^n_\lambda \mathcal{F}(\lambda) \mid \lambda \in \Sigma_\epsilon\}) \leq C_N M_1 \beta_0 \quad (\ell = 0, 1). \tag{3.10}$$

We choose $M_1 \in (0, 1]$ in such a way that $C_N M_1 \beta_0 \leq 1/2$. Since $\|F_\lambda(F, G, H_n)\|_{L_q(\mathbb{R}^N_+)} = \|(F, \lambda^{-1/2}G, H_n, \lambda^{-1/2}\nabla H_n)\|_{L_q(\mathbb{R}^N_+)}$ give us equivalent norms on $X_q(\mathbb{R}^N_+)$ for $\lambda \neq 0$, by (3.10) we see that $(I + F^n_\lambda \mathcal{F}(\lambda))^{-1}$ exists for any $\lambda \in \Sigma_\epsilon$, and therefore

$$v = \mathcal{S}_n(\lambda) F^n_\lambda (I + \mathcal{F}(\lambda)F^n_\lambda)^{-1}(F, G, H_n)$$

is a unique solution to (3.8). Moreover, we have $F^n_\lambda(I + \mathcal{F}(\lambda)F^n_\lambda)^{-1} = (I + F^n_\lambda\mathcal{F}(\lambda))^{-1}F^n_\lambda$. Therefore, setting $\mathcal{S}_{bn}(\lambda) = \mathcal{S}_n(\lambda)(I + F^n_\lambda\mathcal{F}(\lambda))^{-1}$, by (3.10) and Theorem 2.5, we have

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\mathbb{R}^N_+), L_q(\mathbb{R}^N_+)^{N+1})}(\{(\lambda\frac{d}{d\lambda})^\ell(\lambda, \nabla)\mathcal{S}_{bn}(\lambda) \mid \lambda \in \Sigma_\epsilon\}) \leq 2\beta_0 \quad (\ell = 0, 1)$$

and the solution v to (3.8) is represented by $v = \mathcal{S}_{bn}(\lambda)F^n_\lambda(F, G, H_n)$. By the change of variable: $x = \Phi^{-1}(y)$ we have the following theorem.

Theorem 3.5. *Let $1 < q < \infty$ and $0 < \epsilon < \pi/2$. Then, there exists a constant M_1 with $0 < M_1 \leq 1$ depending on q, N and ϵ such that if the condition (3.1) holds, then the following assertion holds: There exists an operator family $\mathcal{T}_n(\lambda) \in \mathrm{Anal}\,(\Sigma_\epsilon, \mathcal{L}(\mathcal{X}_{qn}(\Omega_+), W^1_q(\Omega_+)))$ such that $u = \mathcal{T}_n(\lambda)F^n_\lambda(f, g, h_n)$ is a unique solution to (3.3) for any $(f, g, h_n) \in X_{qn}(\Omega_+)$ and $\lambda \in \Sigma_\epsilon$, and*

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_{qn}(\Omega_+), L_q(\Omega_+)^{N+1})}(\{(\lambda\frac{d}{d\lambda})^\ell(\lambda^{1/2}, \nabla)\mathcal{T}_n(\lambda) \mid \lambda \in \Sigma_\epsilon\}) \leq \beta_1$$

with some constant β_1 depending on β_0, q, ϵ and N.

4 A proof of Theorem 1.5

First, we state some properties of uniform C^1 domain.

Proposition 4.1. *Let Ω be a uniform C^1 domain in $\mathbb{R}^N$. Let M_1 be a positive number given in Theorem 3.4 and Theorem 3.5. Then, there exists positive constants d_i $(i = 0, 1, 2)$ and c_0, at most countably many N-vector of functions $\Phi^i_j \in C^1(\mathbb{R}^N)$ $(i = 1, 2)$ and points $x^0_j \in \Omega$, $x^1_j \in \Gamma_1$ and $x^2_j \in \Gamma_2$ such that the following assertions hold:*

(i) *The map: $\mathbb{R}^N \ni x \mapsto \Phi^i_j(x) \in \mathbb{R}^N$ $(i = 1, 2)$ are bijective.*

(ii) $\Omega = (\bigcup_{j-1}^\infty B_{d^0}(x^0_j)) \cup (\bigcup_{i=1}^2 \bigcup_{j=1}^\infty (\Phi^i_j(\mathbb{R}^N_+) \cap B_{d^i}(d^i_j)))$, $B_{d^0}(x^0_j) \subset \Omega$,
$\Phi^i_j(\mathbb{R}^N_+) \cap B_{d^i}(x^i_j) = \Omega \cap B_{d^i}(x^i_j)$, $\Phi^i_j(\mathbb{R}^N_0) \cap B_{d^i}(x^i_j) = \Gamma_i \cap B_{d^i}(x^i_j)$ $(i = 1, 2)$.

(iii) *There exist C^∞ functions ζ^i_j and $\tilde\zeta^i_j$ such that $0 \le \zeta^i_j, \tilde\zeta^i_j \le 1$, $\operatorname{supp}\zeta^i_j, \operatorname{supp}\tilde\zeta^i_j \subset B_{d^i}(x^i_j)$, $\|\zeta^i_j\|_{W^1_\infty(\mathbb{R}^N)}$, $\|\tilde\zeta^i_j\|_{W^1_\infty(\mathbb{R}^N)} \le c_0$, $\tilde\zeta^i_j = 1$ on $\operatorname{supp}\zeta^i_j$, $\sum_{i=0}^2 \sum_{j=1}^\infty \zeta^i_j = 1$ on $\overline{\Omega}$, and $\sum_{j=1}^\infty \zeta^i_j = 1$ on Γ_i $(i = 1, 2)$.*

(iv) *For $i = 1, 2$ and $j \in \mathbb{N}$, $\nabla\Phi^i_j = \mathcal{A}^i_j + B^i_j(x)$, $\nabla(\Phi^i_j)^{-1} = \mathcal{A}^i_{j,-} + B^i_{j,-}$, where $\mathcal{A}^i_j$ and $\mathcal{A}^i_{j,-}$ are $N \times N$ constant orthonormal matrices, and B^i_j and $B^i_{j,-}$ are $N \times N$ matrices of continous functions defined on $\mathbb{R}^N$ such that $\|(B^i_j, B^i_{j,-})\|_{L_\infty(\mathbb{R}^N)} \le M_1$.*

(v) *There exists a natural number $L \ge 2$ such that any $L + 1$ distinct sets of $\{B_{d^i}(x^i_j) \mid i = 0, 1, 2,\ j \in \mathbb{N}\}$ have an empty intersection.*

In the following, we write $B^i_j = B_{d^i}(x^i_j)$ for the sake of simplicity. By the finite intersection property stated in Proposition 4.1 (v) for any $r \in [1, \infty)$ there exists a constant $C_{r,L}$ such that

$$[\sum_{i=0}^2 \sum_{j=1}^\infty \|f\|_{L_r(\Omega \cap B^i_j)}]^{1/r} \le C_{r,L}\|f\|_{L_r(\Omega)}.$$

The following propositions were proved in Shibata [7, 8].

Proposition 4.2. *Let $1 < q < \infty$, $q' = q/(q-1)$ and $i = 0, 1, 2$. Then, the following assertions hold.*

(i) *Let $\{f_j\}_{j=1}^\infty$ be a sequence in $L_q(\Omega)$ and let $\{g_j\}_{j=1}^\infty$ be a sequence of positive real numbers. Assume that*

$$\sum_{j=1}^\infty g_j^q < \infty \quad \textit{and} \quad |(f_j, \varphi)_\Omega| \le M_3 g_j \|\varphi\|_{L_q(\Omega \cap B^i_j)} \quad \textit{for any } \varphi \in L_{q'}(\Omega) \tag{4.1}$$

with some constant M_3 independent of $j = 1, 2, 3, \ldots$. Then, $f = \sum_{j=1}^\infty f_j$ exists in the strong topology of $L_q(\Omega)$, $(f, \varphi)_\Omega = \sum_{j=1}^\infty (f_j, \varphi)_\Omega$ for any $\varphi \in L_{q'}(\Omega)$, and

$$\|f\|_{L_q(\Omega)} \le C_q M_3 \Big(\sum_{j=1}^\infty g_j^q\Big)^{\frac{1}{q}}.$$

(ii) *Let $\{f_j\}_{j=1}^\infty$ be a sequence in $W^1_q(\Omega)$ such that $\sum_{j=1}^\infty \|f_j\|^q_{W^1_q(\Omega)} < \infty$ and*

$$|(f_j, \varphi)_\Omega| \le M_3 \|f_j\|_{L_q(\Omega)} \|\varphi\|_{L_{q'}(\Omega \cap B^i_j)}, \quad |(D_\ell f_j, \varphi)_\Omega| \le M_3 \|D_\ell f_j\|_{L_q(\Omega)} \|\varphi\|_{L_{q'}(\Omega \cap B^i_j)}$$

for any $\varphi \in L_{q'}(\Omega)$ and $\ell = 1, \ldots, N$. Then, $f = \sum_{j=1}^\infty f_j$ exists in the strong topology of $W^1_q(\Omega)$ with

$$\|f\|_{L_q(\Omega)} \le C_q M_3 \Big(\sum_{j=1}^\infty \|f_j\|^q_{L_q(\Omega)}\Big)^{\frac{1}{q}}, \quad \|\nabla f\|_{L_q(\Omega)} \le C_q M_3 \Big(\sum_{j=1}^\infty \|\nabla f_j\|^q_{L_q(\Omega)}\Big)^{\frac{1}{q}}.$$

(iii) *Let* $\{f_j^{(i)}\}_{j=1}^\infty$ $(i=1,2)$ *be sequences in* $L_q(\Omega)$ *and let* $\{g_j^{(i)}\}_{j=1}^\infty$ $(i=1,2)$ *be sequences of positive numbers. Let* a *and* b *be any complex numbers. Assume that the condition* (4.1) *is satisfied with* $f_j = f_j^{(i)}$ *and* $g_j = g_j^{(i)}$. *In addition, we assume that*

$$|(af_j^{(1)} + bf_j^{(2)}, \varphi)_\Omega| \leq M_3 g_j^{(3)} \|\varphi\|_{L_{q'}(\Omega\cap B_j^i)}$$

with some sequence $\{g_j^{(3)}\}_{j=1}^\infty$ *of positive numbers satisfying condition:* $\sum_{j=1}^\infty (g_j^{(3)})^q < \infty$. *Then,*

$$af^{(1)} + bf^{(2)} = \sum_{j=1}^\infty (af_j^{(1)} + bf_j^{(2)}) \in L_q(\Omega),$$
$$\|af^{(1)} + bf^{(2)}\|_{L_q(\Omega)} \leq C_q M_3 \Big(\sum_{j=1}^\infty (g_j^{(3)})^q\Big)^{\frac{1}{q}}.$$

In the following, we write $\mathcal{H}_j^0 = \mathbb{R}^N$, $\mathcal{H}_j^i = \Phi_j^i(\mathbb{R}_+^N)$, $\partial\mathcal{H}_j^i = \Phi_j^i(\mathbb{R}_0^N)$ $(i=1,2)$ for the sake of simplicity. The following proposition is used to define the infinite sum of $\mathcal{R}$-bounded operators defined on $\mathcal{H}_j^i$.

Proposition 4.3. *Let* $1 < q < \infty$, $q' = q/(q-1)$ *and* $i = 0,1,2$. *Let* Λ *be a domain in* $\mathbb{C}$. *Then, the following assertions hold.*

(i) *Let* $\mathcal{F}(\lambda)$ $(\lambda \in \Lambda)$ *be an operator family in* $\mathcal{L}(L_q(\mathcal{H}_j^i))$ *and let* $\mathcal{G}_k(\lambda)$ $(k=1,\ldots,K)$ *be operator families in* $\mathrm{Anal}\,(\Lambda, \mathcal{L}(L_q(\mathcal{H}_j^i)))$. *Assume that there exist constants* M_4 *and* $M_{5,k}$ *independent of* $j = 1,2,3,\ldots$ *such that*

$$|(\sum_{\ell=1}^n a_\ell \mathcal{F}(\lambda_\ell) f_\ell, \varphi)_{\mathcal{H}_j^i}| \leq M_4 (\sum_{k=1}^K \|\sum_{\ell=1}^n a_\ell \mathcal{G}_k(\lambda_\ell) f_\ell\|_{L_q(\mathcal{H}_j^i)}) \|\varphi\|_{L_q(\mathcal{H}_j^i)},$$
$$\mathcal{R}_{\mathcal{L}(L_q(\mathcal{H}_j^i))}(\{(\lambda \frac{d}{d\lambda})^\ell \mathcal{G}_k(\lambda) \mid \lambda \in \Lambda\}) \leq M_{5,k} \quad (\ell = 0,1, \ k=1,\ldots,K)$$

for any $\varphi \in L_{q'}(\mathcal{H}_j^i)$ *and for any integer* n, $\{a_\ell\}_{\ell=1}^n \subset \mathbb{C}$, $\{\lambda_\ell\}_{\ell=1}^n \subset \Lambda$ *and* $\{f_\ell\}_{\ell=1}^n \subset L_q(\mathcal{H}_j^i)$. *Then,* $\mathcal{F}(\lambda) \in \mathrm{Anal}\,(\Lambda, \mathcal{L}(L_q(\mathcal{H}_j^i)))$ *and*

$$\mathcal{R}_{\mathcal{L}(L_q(\mathcal{H}_j^i))}(\{(\lambda \frac{d}{d\lambda})^\ell \mathcal{F}(\lambda) \mid \lambda \in \Lambda\}) \leq C_q M_4 (\sum_{k=1}^K M_{5,k}^q)^{1/q} \quad (\ell = 0,1).$$

(ii) *Let* $\{\mathcal{F}_j(\lambda)\}_{j=1}^\infty$ *be a sequence in* $\mathrm{Anal}\,(\Lambda, \mathcal{L}(L_q(\mathcal{H}_j^i), L_q(\Omega)))$ *and let* $\{\mathcal{G}_{jk}(\lambda)\}_{j=1}^\infty$ $(k = 1,\ldots,K)$ *be sequences in* $\mathrm{Anal}\,(\Lambda, \mathcal{L}(L_q(\mathcal{H}_j^i)))$. *Assume that there exist constants* M_6 *and* $M_{7,K}$ *independent of* $j = 1,2,3\ldots$ *such that*

$$\mathcal{R}_{\mathcal{L}(L_q(\mathcal{H}_j^i))}(\{(\lambda \frac{d}{d\lambda})^\ell \mathcal{G}_{jk}(\lambda) \mid \lambda \in \Lambda\}) \leq M_{7,k} \quad (\ell = 0,1, \ k=1,\ldots,K),$$
$$|(\sum_{\ell=1}^n a_\ell \mathcal{F}_j(\lambda_\ell) f_\ell, \varphi)_\Omega| \leq M_6 (\sum_{k=1}^K \|\sum_{\ell=1}^n a_\ell \mathcal{G}_{jk}(\lambda_\ell) f_\ell\|_{L_q(\mathcal{H}_j^i)}) \|\varphi\|_{L_{q'}(\Omega\cap B_j^i)}$$

for any integer n, $\{a_\ell\}_{\ell=1}^n \subset \mathbb{C}$, $\{\lambda_\ell\}_{\ell=1}^n \subset \Lambda$ *and* $\{f_\ell\}_{\ell=1}^n \subset L_q(\mathcal{H}_j^i)$ *and for any* $\varphi \in L_{q'}(\Omega)$. *Let* θ_j^i *be operators in* $\mathcal{L}(L_q(\Omega), L_q(\mathcal{H}_j^i))$ $(j = 1,2,3,\ldots)$ *such that* $\|\theta_j^i f\|_{L_q(\mathcal{H}_j^i)} \leq$

$M_8\|f\|_{L_q(\Omega\cap B^i_j)}$ with some constant M_8 for any $f \in L_q(\Omega)$. Then, there exists an operator $\mathcal{F}(\lambda) \in$ Anal $(\Lambda, \mathcal{L}(L_q(\Omega), L_q(\Omega)))$ such that $\mathcal{F}(\lambda)f = \sum_{j=1}^\infty \mathcal{F}_j(\lambda)\theta^i_j f$ in the strong topology of $L_q(\Omega)$ for any $f \in L_q(\Omega)$ and

$$\mathcal{R}_{\mathcal{L}(L_q(\Omega),L_q(\Omega))}(\{(\lambda\frac{d}{d\lambda})^\ell \mathcal{F}(\lambda) \mid \lambda \in \Lambda\}) \leq C_q M_6(\sum_{k=1}^K M_{7,k}^q)^{1/q} M_8 \quad (\ell = 0, 1).$$

Using Theorem 2.2, Theorem 3.4 and Theorem 3.5, we construct a parametrix. For $f \in L_q(\Omega)^N$, $g \in L_q(\Omega)$, $h_d \in W^1_q(\Omega)$ and $h_n \in W^1_q(\Omega)$, let $u^i_j \in W^1_q(\mathcal{H}^i_j)$ be solutions to the following variational equations:

$$(\lambda u^0_j, \varphi)_{\mathcal{H}^0_j} + (\nabla u^0_j, \nabla\varphi)_{\mathcal{H}^0_j} = -(\tilde\zeta^0_j f, \nabla\varphi)_{\mathcal{H}^0_j} + (\tilde\zeta^0_j g, \varphi)_{\mathcal{H}^0_j} \quad \text{for any } \varphi \in W^1_{q'}(\mathcal{H}^0_j), \tag{4.2}$$

$$(\lambda u^1_j, \varphi)_{\mathcal{H}^1_j} + (\nabla u^1_j, \nabla\varphi)_{\mathcal{H}^1_j} = -(\tilde\zeta^1_j f, \nabla\varphi)_{\mathcal{H}^1_j} + (\tilde\zeta^1_j g, \varphi)_{\mathcal{H}^1_j} \quad \text{for any } \varphi \in W^1_{q',0}(\mathcal{H}^1_j) \tag{4.3}$$

subject to $u^1_j = \tilde\zeta^1_j h_d$ on $\partial\mathcal{H}^1_j$, and

$$(\lambda u^2_j, \varphi)_{\mathcal{H}^2_j} + (\nabla u^2_j, \nabla\varphi)_{\mathcal{H}^2_j} = -(\tilde\zeta^2_j f, \nabla\varphi)_{\mathcal{H}^2_j} + (\tilde\zeta^2_j g, \varphi)_{\mathcal{H}^2_j} + < \tilde\zeta^2_j h_n, \varphi >_{\partial\mathcal{H}^2_j} \tag{4.4}$$

for any $\varphi \in W^1_{q'}(\mathcal{H}^2_j)$. By Theorem 2.2, Theorem 3.4 and Theorem 3.5 there exist operator families $T^0_j(\lambda) \in$ Anal $(\Sigma_\epsilon, \mathcal{L}(\mathcal{X}_{q0}(\mathcal{H}^0_j), W^1_q(\mathcal{H}^0_j)))$, $T^1_j(\lambda) \in$ Anal $(\Sigma_\epsilon, \mathcal{L}(\mathcal{X}_{qd}(\mathcal{H}^1_j), W^1_q(\mathcal{H}^1_j)))$ and $T^2_j(\lambda) \in$ Anal $(\Sigma_\epsilon, \mathcal{L}(\mathcal{X}_{qn}(\mathcal{H}^2_j), W^1_q(\mathcal{H}^2_j)))$ such that

$$u^0_j = T^0_j(\lambda)(\tilde\zeta^0_j f, \lambda^{-1/2}\tilde\zeta^0_j g),$$

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_{q0}(\mathcal{H}^0_j), L_q(\mathcal{H}^0_j)^{N+1})}(\{(\lambda\frac{d}{d\lambda})^\ell(\lambda^{1/2}, \nabla)T^0_j(\lambda) \mid \lambda \in \Sigma_\epsilon\}) \leq \beta_2 \quad (\ell = 0, 1), \tag{4.5}$$

$$u^1_j = T^1_j(\lambda)F^d_\lambda(\tilde\zeta^1_j f, \tilde\zeta^1_j g, \tilde\zeta^1_j h_d),$$

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_{qd}(\mathcal{H}^1_j), L_q(\mathcal{H}^1_j)^{N+1})}(\{(\lambda\frac{d}{d\lambda})^\ell(\lambda^{1/2}, \nabla)T^1_j(\lambda) \mid \lambda \in \Sigma_\epsilon\}) \leq \beta_2 \quad (\ell = 0, 1), \tag{4.6}$$

$$u^2_j = T^2_j(\lambda)F^n_\lambda(\tilde\zeta^2_j f, \tilde\zeta^2_j g, \tilde\zeta^2_j h_n),$$

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_{qn}(\mathcal{H}^2_j), L_q(\mathcal{H}^2_j)^{N+1})}(\{(\lambda\frac{d}{d\lambda})^\ell(\lambda^{1/2}, \nabla)T^2_j(\lambda) \mid \lambda \in \Sigma_\epsilon\}) \leq \beta_2 \quad (\ell = 0, 1) \tag{4.7}$$

with some constant β_2 independent of i and j. Set $u = \sum_{i=0}^2\sum_{j=1}^\infty \zeta^i_j u^i_j$. Noting that the $\mathcal{R}$-boundedness implies the boundedness, by (4.5), (4.6) and (4.7) we have

$$\|(\lambda^{1/2}u^i_j, \nabla u^i_j)\|_{L_q(\mathcal{H}^i_j)} \leq \beta_2 \begin{cases} \|(\tilde\zeta^0_j f, \lambda^{-1/2}\tilde\zeta^0_j g)\|_{L_q(\mathcal{H}^0_j)}, \\ \|(\tilde\zeta^1_j f, \lambda^{-1/2}\tilde\zeta^1_j g, \lambda^{1/2}\tilde\zeta^1_j h_d, \nabla(\tilde\zeta^1_j h_d))\|_{L_q(\mathcal{H}^1_j)}, \\ \|(\tilde\zeta^2_j f, \lambda^{-1/2}\tilde\zeta^2_j g, \tilde\zeta^2_j h_n, \lambda^{-1/2}\nabla(\tilde\zeta^2_j h_n))\|_{L_q(\mathcal{H}^2_j)}. \end{cases}$$

By Proposition 4.2, we have $u \in W^1_q(\Omega)$ and

$$\|(\lambda^{1/2}u, \nabla u)\|_{L_q(\Omega)} \leq C_{q,\Omega}\beta_2\|(f, \lambda^{-1/2}g, \lambda^{1/2}h_d, \nabla h_d, h_n, \lambda^{-1/2}\nabla h_n)\|_{L_q(\Omega)}$$

for any $\lambda \in \Sigma_{\epsilon,1}$. Using Proposition 4.2 and noting that $\tilde\zeta^i_j = 1$ on supp ζ^i_j, by (4.2) – (4.7) we have for any $\varphi \in W^1_{q,\Gamma_1}(\Omega)$

$$(\lambda u, \varphi)_\Omega + (\nabla u, \nabla\varphi)_\Omega = \sum_{i=0}^2\sum_{j=1}^\infty[(\lambda\zeta^i_j u^i_j, \varphi)_{\mathcal{H}^i_j} + (\zeta^i_j\nabla u^i_j, \nabla\varphi)_{\mathcal{H}^i_j} + ((\nabla\zeta^i_j)u^i_j, \nabla\varphi)_{\mathcal{H}^i_j}]$$

$$
\begin{aligned}
&= \sum_{i=0}^{2}\sum_{j=1}^{\infty}[(\lambda u_j^i, \zeta_j^i\varphi)_{\mathcal{H}_j^i} + (\nabla u_j^i, \nabla(\zeta_j^i\varphi))_{\mathcal{H}_j^i} - ((\nabla\zeta_j^i)\cdot\nabla u_j^i, \varphi)_{\mathcal{H}_j^i} - (\mathrm{div}\,((\nabla\zeta_j^i)u_j^i), \varphi)_{\mathcal{H}_j^i}] \\
&+ \sum_{j=1}^{\infty} < (\nabla\zeta_j^2)u_j^2, \varphi >_{\partial\mathcal{H}_j^2} \\
&= \sum_{i=0}^{2}\sum_{j=1}^{\infty}[-(f, \nabla(\zeta_j^i\varphi))_{\mathcal{H}_j^i} + (g, \zeta_j^i\varphi)_{\mathcal{H}_j^i} - (2(\nabla\zeta_j^i)\cdot(\nabla u_j^i) + (\Delta\zeta_j^i)u_j^i, \varphi)_{\mathcal{H}_j^i} \\
&+ \sum_{j=1}^{\infty} < \zeta_j^2 h_n + (\nabla\zeta_j^2)u_j^2, \varphi >_{\partial\mathcal{H}_j^2} \\
&= -(f, \nabla\varphi)_\Omega + (g + R_1(f, g, h_d, h_n), \varphi)_\Omega + < h_n + R_2(f, g, h_d, h_n), \varphi >_{\Gamma_2}
\end{aligned}
$$

where we have set

$$
R_1(f, g, h_d, h_n) = -\sum_{i=0}^{2}\sum_{j=1}^{\infty}\{2(\nabla\zeta_j^i)\cdot(\nabla u_j^i) + (\Delta\zeta_j^i)u_j^i\}, \quad R_2(f, g, h_d, h_n) = \sum_{j=1}^{\infty}(\nabla\zeta_j^2)u_j^2. \tag{4.8}
$$

Noting that

$$
\nabla(\zeta_j^1 h_d) = \zeta_j^1\nabla h_d + (\nabla\zeta_j^1)h_d, \quad \lambda^{-1/2}\nabla(\zeta_j^1 h_d) = \zeta_j^1(\lambda^{-1/2}\nabla h_d) + \lambda^{-1/2}(\nabla\zeta_j^1)h_d,
$$

for $F = (F_1, F_2, F_3, F_4, F_5, F_6) \in \mathcal{X}_q(\Omega)$ we define an operator $U(\lambda)$ by

$$
\begin{aligned}
U(\lambda)F &= \sum_{j=1}^{\infty}\zeta_j^0 T_j^0(\tilde\zeta_j^0 F_1, \tilde\zeta_j^0 F_2) \\
&+ \sum_{j=1}^{\infty}\zeta_j^1\{T_j^1(\lambda)(\tilde\zeta_j^1 F_1, \tilde\zeta_j^1 F_2, \tilde\zeta_j^1 F_3, \tilde\zeta_j^1 F_4) + \lambda^{-1/2}T_j^1(\lambda)(0, 0, 0, (\nabla\tilde\zeta_j^1)F_3)\} \\
&+ \sum_{j=1}^{\infty}\zeta_j^2\{T_j^1(\lambda)(\tilde\zeta_j^2 F_1, \tilde\zeta_j^2 F_2, \tilde\zeta_j^2 F_5, \tilde\zeta_j^2 F_6) + \lambda^{-1/2}T_j^2(\lambda)(0, 0, 0, (\nabla\tilde\zeta_j^2)F_5)\}
\end{aligned}
$$

By Proposition 4.3 and (4.5) – (4.7), we have

$$
\begin{aligned}
&u = U(\lambda)F_\lambda(f, g, h_d, h_n), \\
&\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega), W_q^1(\Omega)^{N+1})}(\{(\lambda\frac{d}{d\lambda})^\ell U(\lambda) \mid \lambda \in \Sigma_{\epsilon,1}\}) \le C_{q,\Omega}\beta_2.
\end{aligned} \tag{4.9}
$$

In view of (4.8), we define operators $V^1(\lambda)$ and $V^2(\lambda)$ as follows:

$$
\begin{aligned}
V_1(\lambda)F &= -\sum_{j=1}^{\infty}\{2(\nabla\zeta_j^0)\cdot(\nabla T_j^0(\lambda)(\tilde\zeta_j^0 F_1, \tilde\zeta_j^0 F_2)) + (\Delta\zeta_j^0)T_j^0(\lambda)(\tilde\zeta_j^0 F_1, \tilde\zeta_j^0 F_2))\} \\
&- 2\sum_{j=1}^{\infty}(\nabla\zeta_j^1)\cdot\{\nabla T_j^1(\lambda)(\tilde\zeta_j^1 F_1, \tilde\zeta_j^1 F_2, \tilde\zeta_j^1 F_3, \tilde\zeta_j^1 F_4) + \lambda^{-1/2}\nabla T_j^1(\lambda)(0, 0, 0, (\nabla\tilde\zeta_j^1)F_3)\} \\
&- \sum_{j=1}^{\infty}(\Delta\zeta_j^1)\{T_j^1(\lambda)(\tilde\zeta_j^1 F_1, \tilde\zeta_j^1 F_2, \tilde\zeta_j^1 F_3, \tilde\zeta_j^1 F_4) + \lambda^{-1/2}T_j^1(\lambda)(0, 0, 0, (\nabla\tilde\zeta_j^1)F_3)\}
\end{aligned}
$$

$$
\begin{aligned}
&- 2\sum_{j=1}^\infty (\nabla\zeta_j^2)\cdot\{\nabla T_j^2(\lambda)(\tilde\zeta_j^2F_1, \tilde\zeta_j^2F_2, \tilde\zeta_j^2F_5, \tilde\zeta_j^2F_6) + \lambda^{-1/2}\nabla T_j^2(\lambda)(0,0,0,(\nabla\tilde\zeta_j^2)F_5)\}\\
&- \sum_{j=1}^\infty (\Delta\zeta_j^2)\{T_j^1(\lambda)(\tilde\zeta_j^1F_1, \tilde\zeta_j^2F_2, \tilde\zeta_j^2F_5, \tilde\zeta_j^2F_6) + \lambda^{-1/2}T_j^2(\lambda)(0,0,0,(\nabla\tilde\zeta_j^2)F_5)\},\\
V_2(\lambda)F = &\sum_{j=1}^\infty (\nabla\zeta_j^2)\{T_j^2(\lambda)(\tilde\zeta_j^2F_1, \tilde\zeta_j^2F_2, \tilde\zeta_j^2F_5, \tilde\zeta_j^2F_6) + \lambda^{-1/2}T_j^2(\lambda)(0,0,0,(\nabla\tilde\zeta_j^2)F_5)\}.
\end{aligned}
$$

By Proposition 4.3, Proposition 3.1, Propsition 3.2 and (4.5)–(4.8), we have

$$
\begin{aligned}
&V_1(\lambda)F_\lambda(f,g,h_d,h_n) = R_1(f,g,h_d,h_n),\\
&\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega), L_q(\Omega))}(\{(\lambda\frac{d}{d\lambda})^\ell\lambda^{-1/2}V_1(\lambda) \mid \lambda\in\Sigma_{\epsilon,\lambda_0}\}) \leq C_{q,\Omega}\lambda_0^{-1/2} \quad (\ell=0,1)\\
&V_2(\lambda)F_\lambda(f,g,h_d,h_n) = R_2(f,g,h_d,h_n),\\
&\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega), L_q(\Omega))}(\{(\lambda\frac{d}{d\lambda})^\ell V_2(\lambda) \mid \lambda\in\Sigma_{\epsilon,\lambda_0}\}) \leq C_{q,\Omega}\lambda_0^{-1/2} \quad (\ell=0,1)\\
&\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega), L_q(\Omega))}(\{(\lambda\frac{d}{d\lambda})^\ell\lambda^{-1/2}\nabla V_2(\lambda) \mid \lambda\in\Sigma_{\epsilon,\lambda_0}\}) \leq C_{q,\Omega}\lambda_0^{-1/2} \quad (\ell=0,1).
\end{aligned}
\tag{4.10}
$$

with some constant $C_{q,\Omega}$ depending solely on q and Ω. If we set $\mathcal{V}(\lambda)F = (0, V_1(\lambda)F, 0, V_2(\lambda)F)$, then by (4.10)

$$
\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega), L_q(\Omega))}(\{(\lambda\frac{d}{d\lambda})^\ell F_\lambda\mathcal{V}(\lambda) \mid \lambda\in\Sigma_{\epsilon,\lambda_0}\}) \leq C_{q,\Omega}\lambda_0^{-1/2} \quad (\ell=0,1). \tag{4.11}
$$

Therefore, choosing λ_0 so large that $C_{q,\Omega}\lambda_0^{-1/2}\leq 1/2$, then $(I+F_\lambda\mathcal{V}(\lambda))^{-1}$ exists and

$$
\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega), L_q(\Omega))}(\{(\lambda\frac{d}{d\lambda})^\ell F_\lambda(I+F_\lambda\mathcal{V}(\lambda))^{-1} \mid \lambda\in\Sigma_{\epsilon,\lambda_0}\}) \leq 2 \quad (\ell=0,1). \tag{4.12}
$$

If we set $R(f,g,h_d,h_n) = (0, R_1(f,g,h_d,h_n), 0, R_2(f,g,h_d,h_n)$, then by (4.11)

$$
\|F_\lambda R(f,g,h_d,h_n)\|_{L_q(\Omega)} \leq (1/2)\|F_\lambda(f,g,h_d,h_n)\|_{L_q(\Omega)}.
$$

Since

$$
\|F_\lambda R(f,g,h_d,h_n)\|_{L_q(\Omega)} = \|(f,\lambda^{-1/2}g, \lambda^{1/2}h_d, \nabla h_d, h_n, \nabla^{-1/2}\nabla h_n)\|_{L_q(\Omega)}
$$

give us equivalent norms on $X_q(\Omega)$ for $\lambda\neq 0$, $(I+R)^{-1}$ exists in $\mathcal{L}(X_q(\Omega)$ for any $\lambda\in\Sigma_{\epsilon,\lambda_0}$, which combined with (4.10) furnishes that $u = U(\lambda)F_\lambda(I+R)^{-1}(f,g,h_d,h_n)$ is a unique solution to (1.3). Here, the uniqueness follows from the existence theorem for the dual problem. By (4.9) and (4.11) $R(f,g,h_d,h_n) = \mathcal{V}(\lambda)F_\lambda(f,g,h_d,h_n)$, and therefore $F_\lambda(I+R)^{-1} = (I+F_\lambda\mathcal{V}(\lambda))^{-1}F_\lambda$, which furnishes that $u = U(\lambda)(I+F_\lambda\mathcal{V}(\lambda))^{-1}F_\lambda(f,g,h_d,h_n)$. Setting $\mathcal{A}(\lambda) = U(\lambda)(I+F_\lambda\mathcal{V}(\lambda))^{-1}$, by (4.10) and (4.12) we have

$$
\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega), L_q(\Omega))}(\{(\lambda\frac{d}{d\lambda})^\ell(\lambda^{1/2},\nabla)\mathcal{A}(\lambda) \mid \lambda\in\Sigma_{\epsilon,\lambda_0}\}) \leq 2C_{q,\Omega}\beta_2 \quad (\ell=0,1),
$$

which completes the proof of Theorem 1.5.

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